

MSc Macroeconomics

Topic 2: The Ramsey Model of Growth

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Setup of the Model

- ▶ The utility function is

$$U(\cdot) = \int_{t=0}^{\infty} e^{-\rho t} u(c(t)) L(t) dt$$

- ▶ The flow budget constraint is

$$\dot{A}(t) = r(t)A(t) + w(t)L(t) - c(t)L(t)$$

- ▶ Exogenous factors

(i) Labor: $\dot{L}(t) = nL(t) \Rightarrow L(t) = L(0)e^{nt}$

(ii) Technology: $\dot{A}(t) = xA(t) \Rightarrow A(t) = A(0)e^{xt}$

Per Capita Asset Accumulation

- ▶ Since $c(t)$ is per capita consumption, define $a(t) = A(t)/L(t)$.
- ▶ Find derivative with respect to time and solve for $\dot{A}(t)$
$$\dot{A}(t) = \dot{a}(t)L(t) + na(t)L(t)$$
- ▶ Substitute into capital accumulation and rewrite to get
$$\dot{a}(t) = r(t)a(t) + w(t) - c(t) - na(t)$$

Intertemporal Budget Constraint

- ▶ Rewrite the flow budget constraint as
$$\dot{a}(t) - (r(t) - n)a(t) = w(t) - c(t)$$
- ▶ The integrating factor for the equation above is
$$l(t) = e^{-(\bar{r}(t) - n)t}$$
, where $\bar{r}(t) = (1/t) \int_0^t r(\tau) d\tau$
- ▶ Multiply both sides of the flow budget constraint by this integrating factor and integrate over t to get
$$a(T)e^{-(\bar{r}(T) - n)T} = a(0) + \int_0^T e^{-(\bar{r}(t) - n)t} (w(t) - c(t)) dt$$
- ▶ To check your solution, remember that
$$\frac{d}{dt} \left[\int_{a(t)}^{b(t)} f(z, t) dz \right] = f(b(t), t) b' - f(a(t), t) a' + \int_{a(t)}^{b(t)} \frac{df(z, t)}{dt} dz$$

Ponzi Games

- ▶ As T approaches infinity

$$\lim_{T \rightarrow \infty} a(T) e^{-(\bar{r}(T)-n)T} = a(0) + \int_0^{\infty} e^{-(\bar{r}(t)-n)t} (w(t) - c(t)) dt$$

- ▶ To rule out Ponzi Games we request that the present value of consumption does not exceed total wealth (the sum of initial assets and the present value of wage income). This requirement implies the following condition

$$\lim_{T \rightarrow \infty} a(T) e^{-(\bar{r}(T)-n)T} \geq 0$$

Household' Problem

- ▶ Thus the household chooses the path of consumption and assets so as to maximize

$$U(0) = L(0) \int_{t=0}^{\infty} e^{(n-\rho)t} u(c(t)) dt$$

subject to the following constraints

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t)$$

$$a(0) = a(0)$$

$$\lim_{T \rightarrow \infty} a(T) e^{-(\bar{r}(T) - n)T} \geq 0$$

Hamiltonian and the FOC

- ▶ The Hamiltonian takes the following form

$$J(\cdot) = L(0)e^{(n-\rho)t}u(c(t)) + v(t)((r(t) - n)a(t) + w(t) - c(t))$$

- ▶ The FOC associated with the problem are

$$\frac{\partial J(\cdot)}{\partial c} = 0 \Rightarrow v(t) = u'(c(t))L(0)e^{(n-\rho)t}$$

$$\frac{\partial J(\cdot)}{\partial a} = -\dot{v}(t) \Rightarrow \dot{v}(t) = -(r(t) - n)v(t)$$

$$\lim_{T \rightarrow \infty} (v(t)a(t)) = 0$$

- ▶ The last equation is the transversality condition. (Does it look like the Kuhn-Tucker complementary-slackness condition associated with the inequality constraint?)

Transversality Condition

- ▶ From the Euler's equation we can find $v(t)$ and use the transversality condition to derive
$$\lim_{T \rightarrow \infty} a(T) e^{-(\bar{r}(T)-n)T} = 0$$
- ▶ Remember that to rule out the Ponzi Games we have required that the present value of consumption does not exceed total wealth.
$$\int_0^{\infty} e^{-(\bar{r}(t)-n)t} (c(t)) dt \leq a(0) + \int_0^{\infty} e^{-(\bar{r}(t)-n)t} (w(t)) dt$$
- ▶ But then the household still can increase its utility by increasing consumption. Which means that at the optimum
$$\int_0^{\infty} e^{-(\bar{r}(t)-n)t} (c(t)) dt = a(0) + \int_0^{\infty} e^{-(\bar{r}(t)-n)t} (w(t)) dt$$
- ▶ The latter requirement implies the transversality condition
$$\lim_{T \rightarrow \infty} a(T) e^{-(\bar{r}(T)-n)T} = 0$$

The Euler Equation

- ▶ From the FOC we derive the Euler equation

$$\rho - \frac{\dot{c}(t)}{c(t)} \frac{u''(c(t))c(t)}{u'(c(t))} = r(t)$$

- ▶ Assume that the utility function is (assumption or a requirement?)

$$u(c(t)) = \frac{c^{1-\theta} - 1}{1-\theta}$$

- ▶ Then the Euler equation is re-written as

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\theta}$$

The Consumption Function

- ▶ Solve for $c(t)$ from the Euler's equation

$$c(t) = c(0)e^{\frac{(\bar{r}(t)-\rho)t}{\theta}}$$

- ▶ Substitute into the budget constraint and solve for $c(0)$ to get

$$c(0) = \mu(0) \left(a(0) + \int_0^{\infty} e^{-(\bar{r}(t)-n)t} (w(t)) dt \right)$$

- ▶ In this model consumption is a function of permanent income as opposed to a Keynesian consumption function (consumption is a function of current income) in the Solow-Swan model.

Firms

- ▶ Firms hire labor and rent capital to produce goods using neoclassical production function augmented with Harod neutral technology

$$Y(t) = F(K(t), A(t)L(t))$$

- ▶ The profits are given by

$$\Pi(t) = Y(t) - w(t)L(t) - r(t)K(t) - \delta K(t)$$

- ▶ Optimality requires equalization of rental prices to marginal products

$$\frac{\partial F(\cdot)}{\partial K} = r(t) + \delta$$

$$\frac{\partial F(\cdot)}{\partial L} = w(t)$$

Equilibrium

- ▶ In equilibrium $K(t) = A(t)$. Then

$$\dot{K}(t) = r(t)K(t) + w(t)L(t) - c(t)L(t)$$

- ▶ By Euler's theorem

$$\frac{\partial F(\cdot)}{\partial K} K(t) + \frac{\partial F(\cdot)}{\partial L} L(t) = F(K(t), A(t)L(t))$$

- ▶ Then using the FOC of firms the capital accumulation equation can be rewritten as

$$\dot{K}(t) = F(K(t), A(t)L(t)) - c(t)L(t) - \delta K(t)$$

Balanced Growth Path

- ▶ Along the balanced growth path all variables grow at a constant rate.
- ▶ Constant growth rate of per capita consumption implies constant $r(t)$ (see the Euler equation). But this implies that the MPK is constant as well (see the FOC of the firm) .

$$\frac{\partial F(\cdot)}{\partial K} = \frac{f(\hat{k}(t))}{\partial \hat{k}(t)} = f'(\hat{k}(t))$$

- ▶ This in turn means that $\hat{k}(t)$ is constant \Rightarrow

$$\dot{\hat{k}}(t) = 0 \Rightarrow \frac{\dot{k}(t)}{k(t)} = x$$

Balanced Growth Path, continued

- ▶ Rewrite capital accumulation equation as

$$\frac{\dot{K}(t)}{K(t)} - F\left(1, \frac{1}{\hat{k}(t)}\right) - \delta = \frac{\dot{c}(t)}{k(t)}$$

- ▶ Since the LHS is constant, then so is the RHS. This implies that per capita consumption and capital grow at the same rate.

- ▶ Thus we have that

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{c}(t)}{c(t)} = x$$

- ▶ And $\hat{k}(t)$ and $\hat{c}(t)$ are constant.

The System

- ▶ Using the definition of $\hat{k}(t)$ and $\hat{c}(t)$ we can re-write the capital accumulation equation

$$\dot{\hat{k}}(t) = \underbrace{f'(\hat{k}(t)) - \hat{c}(t)}_{\text{Actual}} - \underbrace{(x + n + \delta)\hat{k}(t)}_{\text{Break-Even}}$$

- ▶ Using the same definitions and the FOC of the firm we can rewrite the Euler equation as

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \frac{f'(\hat{k}(t)) - \delta - \rho - \theta x}{\theta}$$

Steady State

- ▶ We already know that in the steady state $\hat{k}(t) = \hat{c}(t) = 0$
- ▶ Then we can solve for capital per efficiency unit of labor from $f'(\hat{k}^*) = \delta + \rho + \theta x$
- ▶ The steady state levels of consumption per efficiency unit of labor is
$$\hat{c}^* = f(\hat{k}^*) - (x + n + \delta)\hat{k}^*$$

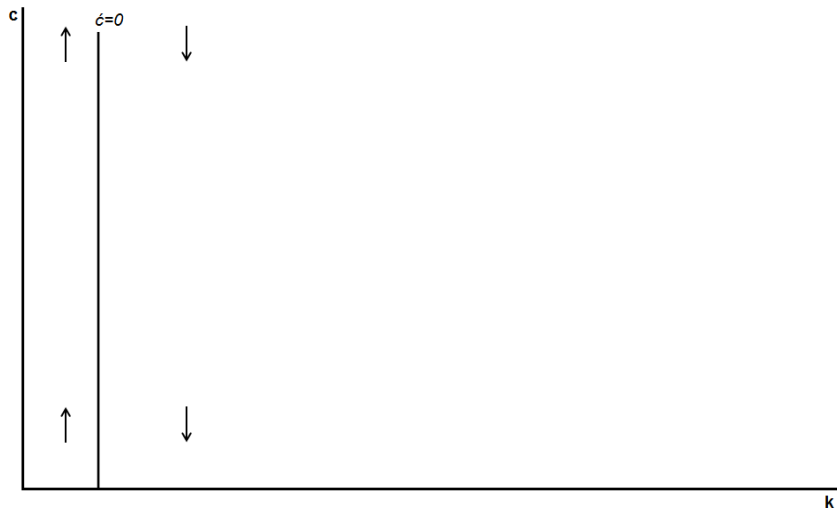
Parameter Restrictions

- ▶ To impose restrictions on parameters so that to ensure bounded utility, find $c(t)$ along the balanced growth path
$$c(t) = c(0)e^{xt}$$
- ▶ Substitute this into the utility function and derive the following restrictions
$$\rho > n + x(1 - \theta)$$

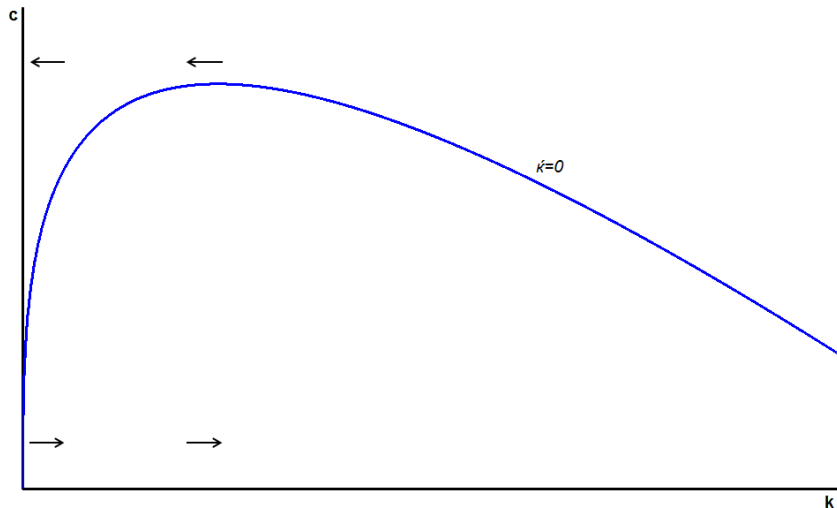
Phase Diagram

- ▶ Phase diagram is used to study qualitative properties of differential equations.
- ▶ We have the following system to plot in $(\hat{c}(t), \hat{k}(t))$ space.
$$f'(\hat{k}(t)) = \delta + \rho + \theta x$$
$$\hat{c}(t) = f(\hat{k}(t)) - (x + n + \delta)\hat{k}(t)$$

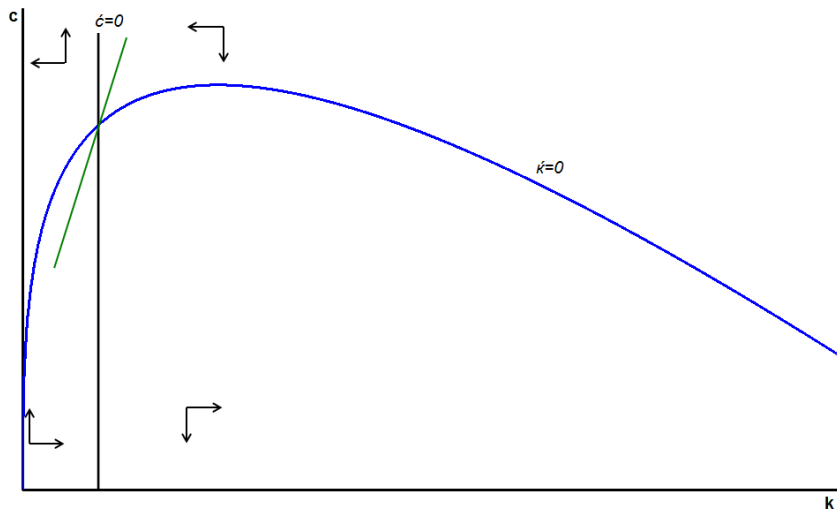
The Dynamics of $\hat{c}(t)$



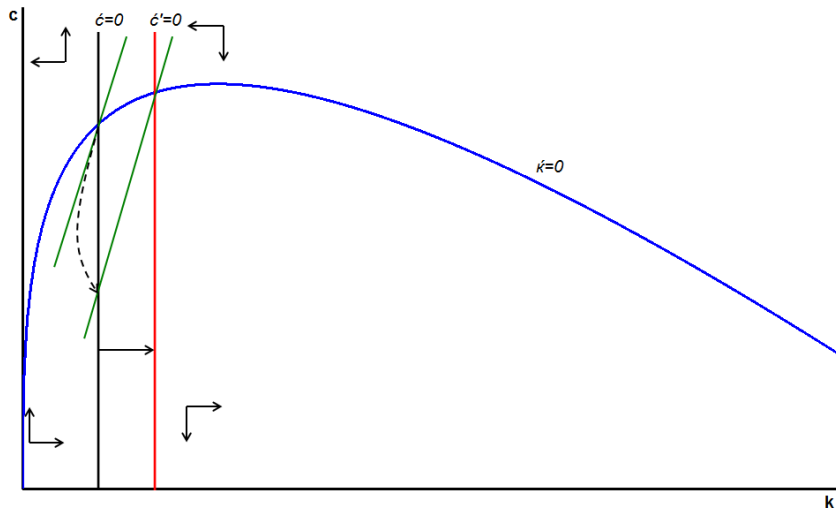
The Dynamics of $\hat{k}(t)$



The Dynamics of $\hat{c}(t)$ and $\hat{k}(t)$



The Effects of the Fall in the Discount Rate



The Effects of Government Purchases

- ▶ Assume that the government buys $G(t)$ units of output expressed in effective labor units. Assume also that this does not affect utility from private consumption. To finance this consumption the government collects lump-sum taxes, $T(t)$.

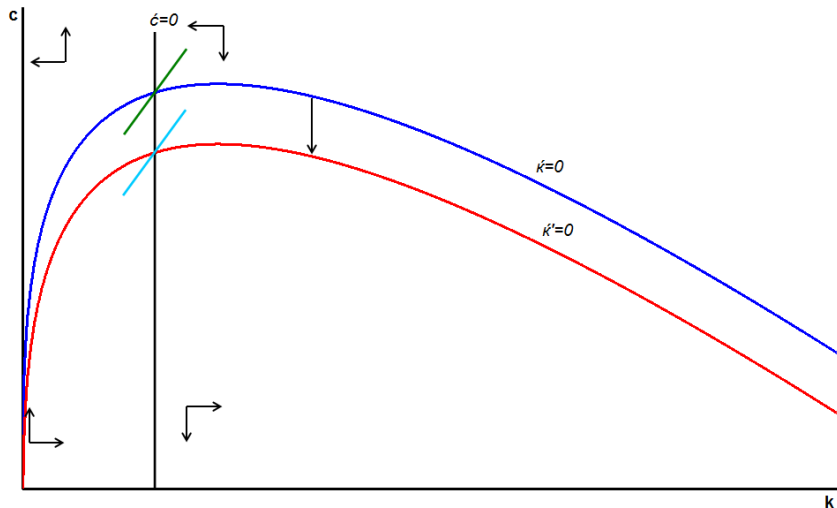
The capital accumulation equation is

$$\dot{\hat{k}}(t) = f(\hat{k}(t)) - \hat{c}(t) - (x + n + \delta)\hat{k}(t) - T(t)$$

- ▶ The government runs balanced budget, $T(t) = G(t) \Rightarrow$

$$\dot{\hat{k}}(t) = f(\hat{k}(t)) - \hat{c}(t) - (x + n + \delta)\hat{k}(t) - G(t)$$

The Effects of Government Purchases



The Effects of Taxes

- ▶ Now assume that the government taxes wage income and asset (capital) income at rates τ_w and τ_a respectively.
- ▶ The government runs balanced budget and the taxes are refunded in form of transfer, $V(t)$.
$$V(t) = \tau_w w(t)L(t) + \tau_a r(t)A(t) \Rightarrow$$
$$v(t) = \tau_w w(t) + \tau_a r(t)a(t).$$
- ▶ The per capita flow budget constraint takes the following form
$$\dot{a}(t) = (1 - \tau_a)r(t)a(t) + (1 - \tau_w)w(t) - c(t) - na(t) + v(t)$$

The Effects of Taxes, continued

- ▶ The firms FOC are

$$r(t) = f'(\hat{k}(t)) - \delta$$

$$w(t) = (f(\hat{k}(t)) - f'(\hat{k}(t))\hat{k}(t))A(t)$$

- ▶ The Euler equation is

$$\frac{\dot{c}(t)}{c(t)} = \frac{(1 - \tau_a)r(t) - \rho}{\theta} \Rightarrow$$

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \frac{(1 - \tau_a)(f'(\hat{k}(t)) - \delta) - \rho - \theta x}{\theta}$$

- ▶ Using the Euler's theorem and the FOCs of firms the flow budget constraint is rewritten as

$$\hat{k}(t) = f(\hat{k}(t)) - \hat{c}(t) - (x + n + \delta)\hat{k}(t)$$

The Effects of Taxes, continued

- ▶ Note that introduction of taxes does not affect the growth rate along the BGP. Per capita variables still grow at the same rate as the technological progress.
- ▶ Taxes have level effect on the steady state variables.

Discussion

- ▶ Microfounded demand side.
- ▶ BGP properties of the model compared to the Solow-Swan model.
- ▶ Transitional dynamics
- ▶ Taxation
- ▶ The model's ability to explain long run growth.