

MSc Macroeconomics

Topic 3: The AK and Romer Models of Growth

Vahagn Galstyan

Trinity College Dublin
v.galstyan@tcd.ie

<http://www.vahagn-galstyan.com/teaching>

2009-2010

Solow Model Revisited

- ▶ Remember the Solow-Swan model?

$$\dot{K} = sY - \delta K$$

- ▶ In per capita terms

$$\dot{k} = sf(k) - (n + \delta)k$$

- ▶ If Y satisfies properties of neoclassical production function then per capita growth rate is given by exogenous rate of growth of technology

Solow Model Rewritten

- ▶ Assume a production function where the marginal product of capital is not declining

$$Y = AK$$

- ▶ Per capita capital accumulation is

$$\frac{\dot{k}}{k} = sA - (n + \delta)$$

- ▶ Note that capital is growing even without exogenous technical progress

$$\frac{\dot{k}}{k} = sA - (n + \delta)$$

Solow Model Rewritten

- ▶ Since $y = Ak$ and $c = (1 - s)y$ then per capita output and consumption grow at the same rate as per capita capital

$$\frac{\dot{k}}{k} = \frac{\dot{c}}{c} = \frac{\dot{y}}{y} = sA - (n + \delta)$$

- ▶ Note that savings, technology, population growth and depreciation have permanent effects on the growth rate of per capita variables
- ▶ Why do we have growth?

AK Model

- ▶ Assume that the production function of a firm is $y_j = \bar{A}k_j^\alpha L_j^{1-\alpha}$ (constant returns at firm level)
- ▶ $\bar{A} = A_0 \left(\sum_{j=1}^N k_j \right)^\eta$
- ▶ $K = \sum_{j=1}^N k_j$ and $Y = \sum_{j=1}^N y_j$
- ▶ In equilibrium $k_j = k = K/N$ therefore $Y = AK^{\alpha+\eta}$
- ▶ Capital accumulation is $\dot{K} = sY - \delta K$. It follows that
$$\frac{\dot{K}}{K} = sAK^{\alpha+\eta-1} - \delta$$

AK Model

- ▶ $\frac{\dot{K}}{K} = sAK^{\alpha+\eta-1} - \delta$
- ▶ For constant long run growth to exist $\alpha + \eta - 1 = 0$. Why?
- ▶ $\alpha + \eta < 1$, knowledge spillovers not strong enough
- ▶ $\alpha + \eta > 1$, knowledge spillovers too strong
- ▶ $\alpha + \eta = 1$, knowledge spillovers just right

AK Model with Utility Maximization

- ▶ The utility function is

$$U(\cdot) = \int_{t=0}^{\infty} e^{-\rho t} \frac{c^{1-\varepsilon}}{1-\varepsilon} dt$$

- ▶ The flow budget constraint is

$$\dot{k} = \bar{A}k^{\alpha} - c - \delta k$$

AK Model with Utility Maximization

- ▶ The Hamiltonian takes the following form

$$H(\cdot) = e^{-\rho t} \frac{c^{1-\varepsilon}}{1-\varepsilon} + \lambda(\bar{A}k^\alpha - c - \delta k)$$

- ▶ The FOC associated with the problem are

$$\frac{\partial H(\cdot)}{\partial c} = 0$$

$$\frac{\partial H(\cdot)}{\partial k} = -\dot{\lambda}$$

- ▶
$$\frac{\dot{c}}{c} = \frac{\alpha A k^{1+\eta-1} - \delta - \rho}{\varepsilon}$$

AK Model with Utility Maximization

- ▶ $\frac{\dot{c}}{c} = \frac{\alpha Ak^{1+\eta-1} - \delta - \rho}{\varepsilon}$
- ▶ $\alpha + \eta < 1$
- ▶ $\alpha + \eta > 1$
- ▶ $\alpha + \eta = 1$

Romer's Approach

- ▶ Representative household maximizes the following utility function

$$U(\cdot) = \int_{t=0}^{\infty} e^{-\rho t} \frac{c^{1-\varepsilon}}{1-\varepsilon} L dt$$

- ▶ Resource constraint is

$$\dot{B} = rB + wL - cL$$

- ▶ Associated Euler's equation is

$$\gamma_c = \frac{\dot{c}}{c} = \frac{r - \rho}{\theta}$$

Final Goods

- ▶ Final good is produced by

$$Y = L^{1-\alpha} \sum_{i=1}^M x_i^\alpha$$

- ▶ Profit is

$$\pi = Y - wL - \sum_{j=1}^M p_j x_j$$

FOC in Final Goods Sector

- ▶ The FOC with respect to labor

$$\frac{\partial \pi}{\partial L} = (1 - \alpha) \frac{Y}{L} - w = 0$$

- ▶ The FOC with respect to intermediate input

$$\frac{\partial \pi}{\partial x_i} = \alpha L^{1-\alpha} x_i^{\alpha-1} - p_i = 0$$

Intermediate Goods Sector

- ▶ Profit of a monopolistic producer of an intermediate good is

$$\pi_i = p_i x_i - x_i = \alpha L^{1-\alpha} x_i^\alpha - x_i$$

- ▶ The FOC

$$\frac{\partial \pi}{\partial x_i} = \alpha^2 L^{1-\alpha} x_i^{\alpha-1} - 1 = 0 \Rightarrow \pi = \frac{1-\alpha}{\alpha} L \alpha^{\frac{2}{1-\alpha}}$$

- ▶ Since $Y = L^{1-\alpha} \sum_{i=1}^M x_i^\alpha = L^{1-\alpha} M x^\alpha$ then the growth rate of Y is the same as the growth rate of M .

The Growth Rate

- ▶ Assume that output of research is the flow of new blueprints

$$\dot{M} = \lambda R_t$$

- ▶ Assume that research sector is perfectly competitive. Then

$$\Pi = \frac{\pi}{r} \lambda R_t - R_i = 0 \Rightarrow r = \lambda \pi$$

- ▶ From Euler's equation

$$\gamma_c = \frac{\dot{c}}{c} = \frac{\lambda \pi - \rho}{\theta} = \frac{\lambda \frac{1-\alpha}{\alpha} L \alpha^{\frac{2}{1-\alpha}} - \rho}{\theta}$$