

# MSc Macroeconomics

## Topic 3: The AK and Romer Models of Growth

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# Solow Model Revisited

- ▶ Remember the Solow-Swan model?

$$\dot{K} = sY - \delta K$$

- ▶ In per capita terms

$$\dot{k} = sf(k) - (n + \delta)k$$

- ▶ If  $Y$  satisfies properties of neoclassical production function then per capita growth rate is given by exogenous rate of growth of technology

# Solow Model Rewritten

- ▶ Assume a production function where the marginal product of capital is not declining

$$Y = AK$$

- ▶ Per capita capital accumulation is

$$\frac{\dot{k}}{k} = sA - (n + \delta)$$

- ▶ Note that capital is growing even without exogenous technical progress

$$\frac{\dot{k}}{k} = sA - (n + \delta)$$

# Solow Model Rewritten

- ▶ Since  $y = Ak$  and  $c = (1 - s)y$  then per capita output and consumption grow at the same rate as per capita capital

$$\frac{\dot{k}}{k} = \frac{\dot{c}}{c} = \frac{\dot{y}}{y} = sA - (n + \delta)$$

- ▶ Note that savings, technology, population growth and depreciation have permanent effects on the growth rate of per capita variables
- ▶ Why do we have growth?

# AK Model

- ▶ Assume that the production function of a firm is  $y_j = \bar{A}k_j^\alpha L_j^{1-\alpha}$  (constant returns at firm level)
- ▶  $\bar{A} = A_0 \left( \sum_{j=1}^N k_j \right)^\eta$
- ▶  $K = \sum_{j=1}^N k_j$  and  $Y = \sum_{j=1}^N y_j$
- ▶ In equilibrium  $k_j = k = K/N$  therefore  $Y = AK^{\alpha+\eta}$
- ▶ Capital accumulation is  $\dot{K} = sY - \delta K$ . It follows that 
$$\frac{\dot{K}}{K} = sAK^{\alpha+\eta-1} - \delta$$

# AK Model

- ▶  $\frac{\dot{K}}{K} = sAK^{\alpha+\eta-1} - \delta$
- ▶ For constant long run growth to exist  $\alpha + \eta - 1 = 0$ . Why?
- ▶  $\alpha + \eta < 1$ , knowledge spillovers not strong enough
- ▶  $\alpha + \eta > 1$ , knowledge spillovers too strong
- ▶  $\alpha + \eta = 1$ , knowledge spillovers just right

# AK Model with Utility Maximization

- ▶ The utility function is

$$U(\cdot) = \int_{t=0}^{\infty} e^{-\rho t} \frac{c^{1-\varepsilon}}{1-\varepsilon} dt$$

- ▶ The flow budget constraint is

$$\dot{k} = \bar{A}k^{\alpha} - c - \delta k$$

# AK Model with Utility Maximization

- ▶ The Hamiltonian takes the following form

$$H(\cdot) = e^{-\rho t} \frac{c^{1-\varepsilon}}{1-\varepsilon} + \lambda(\bar{A}k^\alpha - c - \delta k)$$

- ▶ The FOC associated with the problem are

$$\frac{\partial H(\cdot)}{\partial c} = 0$$

$$\frac{\partial H(\cdot)}{\partial k} = -\dot{\lambda}$$

- ▶ 
$$\frac{\dot{c}}{c} = \frac{\alpha A k^{\alpha+\eta-1} - \delta - \rho}{\varepsilon}$$

# AK Model with Utility Maximization

$$\blacktriangleright \frac{\dot{c}}{c} = \frac{\alpha A k^{\alpha+\eta-1} - \delta - \rho}{\varepsilon}$$

$$\blacktriangleright \alpha + \eta < 1$$

$$\blacktriangleright \alpha + \eta > 1$$

$$\blacktriangleright \alpha + \eta = 1$$

# AK Model: Discussion

- ▶ Distinction between capital accumulation and technological progress
- ▶ Convergence
- ▶ Acemoglu and Ventura (2002)

# Romer's Approach

- ▶ Representative household maximizes the following utility function

$$U(\cdot) = \int_{t=0}^{\infty} e^{-\rho t} \frac{c^{1-\varepsilon}}{1-\varepsilon} L dt$$

- ▶ Resource constraint is

$$\dot{B} = rB + wL - cL$$

- ▶ Associated Euler's equation is

$$\gamma_c = \frac{\dot{c}}{c} = \frac{r - \rho}{\theta}$$

# Final Goods

- ▶ Final good is produced by

$$Y = L^{1-\alpha} \sum_{i=1}^M x_i^\alpha$$

- ▶ Profit is

$$\pi = Y - wL - \sum_{j=1}^M p_j x_j$$

# FOC in Final Goods Sector

- ▶ The FOC with respect to labor

$$\frac{\partial \pi}{\partial L} = (1 - \alpha) \frac{Y}{L} - w = 0$$

- ▶ The FOC with respect to intermediate input

$$\frac{\partial \pi}{\partial x_i} = \alpha L^{1-\alpha} x_i^{\alpha-1} - p_i = 0$$

# Intermediate Goods Sector

- ▶ Profit of a monopolistic producer of an intermediate good is

$$\pi_i = p_i x_i - x_i = \alpha L^{1-\alpha} x_i^\alpha - x_i$$

- ▶ The FOC

$$\frac{\partial \pi}{\partial x_i} = \alpha^2 L^{1-\alpha} x_i^{\alpha-1} - 1 = 0 \Rightarrow \pi = \frac{1-\alpha}{\alpha} L \alpha^{\frac{2}{1-\alpha}}$$

- ▶ Since  $Y = L^{1-\alpha} \sum_{i=1}^M x_i^\alpha = L^{1-\alpha} M x^\alpha$  then the growth rate of  $Y$  is the same as the growth rate of  $M$ .

# The Growth Rate

- ▶ Assume that output of research is the flow of new blueprints

$$\dot{M} = \lambda R_t$$

- ▶ Assume that research sector is perfectly competitive. Then

$$\Pi = \frac{\pi}{r} \lambda R_t - R_i = 0 \Rightarrow r = \lambda \pi$$

- ▶ From Euler's equation

$$\gamma_c = \frac{\dot{c}}{c} = \frac{\lambda \pi - \rho}{\theta} = \frac{\lambda \frac{1-\alpha}{\alpha} L \alpha^{\frac{2}{1-\alpha}} - \rho}{\theta}$$

# Romer's Approach: The Results

- ▶ The growth decreases with the rate of time preference
- ▶ The growth increases with the productivity of research
- ▶ The growth increases with the size of the economy (Jones points to this prediction as counterfactual, since the number of researchers in the US has increased, but the growth rate has remained constant.)